THE MATHEMATICAL ASSOCIATION OF AMERICA
AMERICAN MATHEMATICS COMPETITIONS

27th Annual

AMERICAN INVITATIONAL MATHEMATICS EXAMINATION

(AIME I)

Tuesday, March17, 2009

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR PROCTOR GIVES THE SIGNAL TO BEGIN.

2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.

3. No aids other than scratch paper, graph paper, ruler, compass, and protractor are permitted. In particular, **calculators and computers are not permitted.**

4. A combination of the AIME and the American Mathematics Contest 10 or the American Mathematics Contest 12 scores are used to determine eligibility for participation in the U.S.A. Mathematical Olympiad (USAMO). The USAMO will be given in your school on TUESDAY and WEDNESDAY, April 28-29, 2009.

5. Record all of your answers, and certain other information, on the AIME answer form. Only the answer form will be collected from you.

1. Call a 3-digit number *geometric* if it has 3 distinct digits which, when read from left to right, form a geometric sequence. Find the difference between the largest and smallest geometric numbers.

2. There is a complex number z with imaginary part 164 and a positive integer n such that
$$\frac{z}{z+n} = 4i.$$
Find n.

3. A coin that comes up heads with probability $p > 0$ and tails with probability $1-p > 0$ independently on each flip is flipped eight times. Suppose the probability of three heads and five tails is equal to $\frac{1}{25}$ of the probability of five heads and three tails. Let $p = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

4. In parallelogram $ABCD$, point M is on $\overline{AB}$ so that $\frac{AM}{AB} = \frac{17}{1000}$, and point N is on $\overline{AD}$ so that $\frac{AN}{AD} = \frac{17}{2009}$. Let P be the point of intersection of $\overline{AC}$ and $\overline{MN}$. Find $\frac{AC}{AP}$.

5. Triangle ABC has $AC = 450$ and $BC = 300$. Points K and L are located on $\overline{AC}$ and $\overline{AB}$ respectively so that $AK = CK$, and $\overline{CL}$ is the angle bisector of angle C. Let P be the point of intersection of $\overline{BK}$ and $\overline{CL}$, and let M be the point on line BK for which K is the midpoint of $\overline{PM}$. If $AM = 180$, find LP.

6. How many positive integers N less than 1000 are there such that the equation $x^{\lfloor x \rfloor} = N$ has a solution for x? (The notation $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x.)

7. The sequence (a_n) satisfies $a_1 = 1$ and $5^{(a_{n+1}-a_n)} - 1 = \dfrac{1}{n+\frac{2}{3}}$ for $n \geq 1$. Let k be the least integer greater than 1 for which a_k is an integer. Find k.

8. Let $S = \{2^0, 2^1, 2^2, \ldots, 2^{10}\}$. Consider all possible positive differences of pairs of elements of S. Let N be the sum of all of these differences. Find the remainder when N is divided by 1000.

9. A game show offers a contestant three prizes A, B and C, each of which is worth a whole number of dollars from \$1 to \$9999 inclusive. The contestant wins the prizes by correctly guessing the price of each prize in the

order A, B, C. As a hint, the digits of the three prices are given. On a particular day, the digits given were 1, 1, 1, 1, 3, 3, 3. Find the total number of possible guesses for all three prizes consistent with the hint.

10. The Annual Interplanetary Mathematics Examination (AIME) is written by a committee of five Martians, five Venusians, and five Earthlings. At meetings, committee members sit at a round table with chairs numbered from 1 to 15 in clockwise order. Committee rules state that a Martian must occupy chair 1 and an Earthling must occupy chair 15. Furthermore, no Earthling can sit immediately to the left of a Martian, no Martian can sit immediately to the left of a Venusian, and no Venusian can sit immediately to the left of an Earthling. The number of possible seating arrangements for the committee is $N \cdot (5!)^3$. Find N.

11. Consider the set of all triangles OPQ where O is the origin and P and Q are distinct points in the plane with nonnegative integer coordinates (x, y) such that $41x + y = 2009$. Find the number of such distinct triangles whose area is a positive integer.

12. In right $\triangle ABC$ with hypotenuse $\overline{AB}$, $AC = 12$, $BC = 35$, and $\overline{CD}$ is the altitude to $\overline{AB}$. Let ω be the circle having $\overline{CD}$ as a diameter. Let I be a point outside $\triangle ABC$ such that $\overline{AI}$ and $\overline{BI}$ are both tangent to circle ω. The ratio of the perimeter of $\triangle ABI$ to the length AB can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

13. The terms of the sequence (a_i) defined by $a_{n+2} = \dfrac{a_n + 2009}{1 + a_{n+1}}$ for $n \geq 1$ are positive integers. Find the minimum possible value of $a_1 + a_2$.

14. For $t = 1, 2, 3, 4$, define $S_t = \sum_{i=1}^{350} a_i^t$, where $a_i \in \{1, 2, 3, 4\}$. If $S_1 = 513$ and $S_4 = 4745$, find the minimum possible value for S_2.

15. In triangle ABC, $AB = 10$, $BC = 14$, and $CA = 16$. Let D be a point in the interior of $\overline{BC}$. Let I_B and I_C denote the incenters of triangles ABD and ACD, respectively. The circumcircles of triangles BI_BD and CI_CD meet at distinct points P and D. The maximum possible area of $\triangle BPC$ can be expressed in the form $a - b\sqrt{c}$, where a, b and c are positive integers and c is not divisible by the square of any prime. Find $a + b + c$.

Your Exam Manager will receive a copy of the 2009 AIME Solution Pamphlet with the scores.

CONTACT US -- *Correspondence about the problems and solutions for this AIME and orders for any of our publications should be addressed to:*

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2009 USAMO -- THE USA MATHEMATICAL OLYMPIAD (USAMO) is a 6-question, 9-hour, essay-type examination. The USAMO will be held in your school on Tuesday and Wednesday, April 28-29, 2009. Your teacher has more details on who qualifies for the USAMO in the AMC 10/12 and AIME Teachers' Manuals. The best way to prepare for the USAMO is to study previous years of these exams, the World Olympiad Problems/Solutions and review the contents of the Arbelos. Copies may be ordered from the web sites indicated below.

PUBLICATIONS -- For a complete listing of available publications please visit the following web sites:

AMC - - http://www.unl.edu/amc/d-publication/publication.shtml
MAA -- https://enterprise.maa.org/ecomtpro/timssnet/common/tnt_frontpage.cfm

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